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SHORTER COMMUNICATIONS

EXACT SOLUTION OF A STEFAN PROBLEM RELEVANT TO CONTINUOUS CASTING

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NOMENCLATURE

c, specific heat of solidified material;
K, thermal conductivity of solidified material;
L', $= Lc(T_0 - T_1)$, latent heat of solidification;
T', temperature: $T = (T' - T_1)/(T_0 - T_1)$;
T_0, temperature of solidification;
T_1, temperature of cooling boundary;
u, v, parabolic cylinder coordinates in normalized problem:

$$x = 2uv,\ y = u^2 - v^2;$$

V, uniform speed of advance of the material in the positive y' direction;
x', y', coordinates in physical plane: $x' = xKV\ \rho c$, $y' = yKV\ \rho c$;
y' $= F(x')$, equation of solidification interface:

$$y = f(x) = V\rho c/K\ F(x').$$

Greek symbol
ρ, density of solidified material.

INTRODUCTION

KNOWN exact solutions of Stefan problems are limited in number. Those found up to 1959 are discussed by [1]. Amongst the fields of application of Stefan problems today continuous casting of metal is prominent and much work has been done, both numerically and analytically.

A recent paper by [2] gives an "exact" 2-dim. solution for the slab-casting problem but the author argues that in many instances heat flow by conduction dominates heat flow by transport of material. Hence he uses Laplace's equation to describe the steady heat flow.

The present treatment is also 2-dim. and since primary interest was in the shape of the solidification interface near the wall of the mold, the liquid and solid material is assumed to occupy the complete right half-plane (and thus, in effect, one wall of the mold is removed to infinity). On the other hand, the full steady heat flow equation is used, i.e. the term describing heat transport is retained, and thus we might expect that the results are more generally applicable.

As in Siegel's case the present analysis is for a so-called "single phase" problem in which the temperature of the incoming liquid material is assumed uniform and equal to the freezing temperature of the material. Density and specific heat of liquid and solid are assumed equal.

The connection between the solution derived for the present problem and that for the growth of a parabolic dendritic platelet by [3] is also discussed briefly below.

FORMULATION OF THE PROBLEM

Description of the 2-dim. model

The model is illustrated in Fig. 1. The material is assumed to advance uniformly in the positive y-direction at constant speed. The temperature distribution in the solid and, in particular, the location of the interface is to be determined.

Boundary conditions

An appropriate condition to describe heat flow at the cooling surface $x' = 0$, $y' > 0$ must be chosen. In order to obtain an exact solution, it is necessary to make this choice the simplest, i.e. constant temperature $T_1 < T_0$ where T_0 is the freezing temperature of the material. (This is also the boundary condition used by [2]).

Such a condition is obviously well removed from the situation prevailing in practical casting but it is hoped that the solution obtained here will be a suitable starting point for approximate analysis of problems with more realistic cooling conditions. The latter might be the so-called "radiation" condition $T'_{x'} = \lambda(T' - T_1)$ where $\lambda = H/K$, H being the coefficient of "surface heat transfer" and K the thermal conductivity of the material. We note that this condition reduces to the one used here as $\lambda \to \infty$.

On the liquid–solid interface, the temperature of liquid and solid are equal. On the interface, we must apply also the Stefan condition for absorption of latent heat on freezing in the appropriate form for a steady state 2-dim. problem with material transport at constant speed. The temperature is assumed bounded everywhere.

The mathematical statement

$$T'_{x'x'} + T'_{y'y'} = \frac{V\rho c}{K} T'_{y'},\ x' > 0:\ y' > F(x')$$

$$x' = 0,\quad T' = T_1,\ y' > 0$$

$$T'_x = 0 \quad F(0) < y' < 0$$

$$y' = F(x') \quad T' = T_0$$

$$T'_{y'} = -\frac{L'\rho V}{K\left[1+\left(\frac{dF}{dx'}\right)^2\right]}.$$

Making the normalizing substitutions

$$T = \frac{T' - T_1}{T_0 - T_1}$$

$$x = \frac{V\rho c}{K}x', \quad y = \frac{V\rho c}{K}y'$$

$$L = \frac{L'}{c(T_0 - T_1)}$$

$$f(x) = \frac{V\rho c}{K}F(x')$$

we obtain

$$T_{xx} + T_{yy} = T_y, \quad x > 0: y > f(x) \tag{1}$$

$$x = 0,\ T = 0 \quad y > 0 \tag{2}$$

$$T_x = 0 \quad f(0) < y < 0 \tag{3}$$

$$y = f(x),\ T = 1 \tag{4}$$

$$T_y = -\frac{L}{1+f'^2} \tag{5}$$

THE SOLUTION

The equation and boundary conditions (1)–(5) permit an exact solution in parabolic cylinder co-ordinates.

Put $x = 2uv$, $y = u^2 - v^2$, then the problem becomes

$$T_{uu} - 2uT_u + T_{vv} + 2vT_v = 0, \quad u > 0: 0 < v < \phi(u) \tag{6}$$

$$v = 0, \quad T = 0, \quad u > 0 \tag{7}$$

$$u = 0, \quad T_u = 0, \quad 0 < v < \phi(0) \tag{8}$$

$$v = \phi(u), \quad T = 1 \tag{9}$$

$$T_v = \frac{2L[\phi + u\phi']}{1+\phi'^2}. \tag{10}$$

If T is independent of u and $\phi(u)$ = constant = v_0, say, the set (6)–(10) reduces to

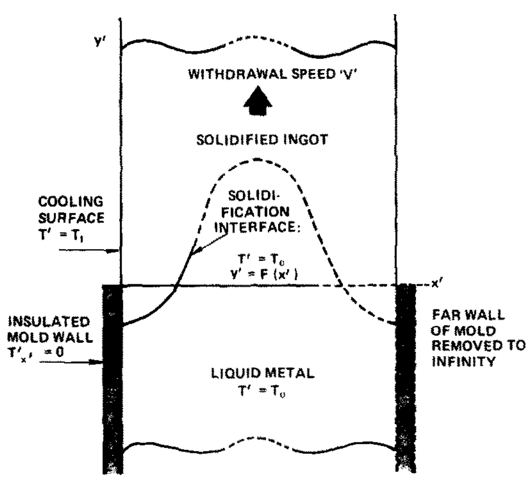

FIG. 1. The 2-dim. model.

$$T_{vv} + 2vT_v = 0, \quad 0 < v < v_0 \tag{11}$$

$$v = 0, \quad T = 0 \tag{12}$$

$$v = v_0, \quad T = 1 \tag{13}$$

$$T_v = 2Lv_0. \tag{14}$$

The equations and boundary conditions (11)–(14) are mathematically equivalent to those of Neumann's classical 1-dim. time-dependent freezing–melting problem [1], if v is replaced by $x/2t^{1/2}$. Hence we obtain

$$T = \frac{\operatorname{erf} v}{\operatorname{erf} v_0} \tag{15}$$

with v_0 determined by

$$L\Pi^{1/2}\, v_0\, e^{v_0^2} \operatorname{erf} v_0 = 1. \tag{16}$$

In rectangular coordinates, the liquid–solid interface $v = v_0$ becomes

$$y = \frac{x^2}{4v_0^2} - v_0^2.$$

After this work was completed, we noted that the transformation to parabolic cylinder coordinates had been used by [3] in the solution of one of a number of Stefan problems in crystal growth. In their case, its use was rather more obvious, since the relevant problem was the growth of a parabolic dendritic platelet advancing into an unbounded liquid medium.

Horvay and Cahn *began* with the parabolic shape of the freezing front, and deduced the temperature distribution in front of, and behind, the parabola.

Our problem, as has been seen, was converse in its reasoning, and complicated by the fact that we were working in a half-plane with boundary conditions to be satisfied on the y-axis. Beginning with a uniform temperature distribution behind a freezing front of unknown shape, we have *deduced* the parabolic equation of the front, and the temperature distribution in front of it.

It is not surprising, in view of the converse approach of [3], that the relationship of their final mathematical results for the parabolic platelet to those of the Neumann problem, necessarily more distant than ours, is not indicated.

COMMENTS

It is hoped that the solution obtained, apart from any intrinsic merit it may have, will provide a basic for analytical approximate solution of models with more sophisticated boundary conditions and/or geometry. Further, if the reduction of the present 2-dim. steady state problem to the mathematically equivalent similarity solution of Neumann is not merely coincidence there is a possibility of further 2-dim. solutions of interest being derived from known similarity solutions of Stefan problems.

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